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LETTER TO THE EDITOR

Gauge invariance as the Lie–Bäcklund transformation group

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Abstract. The gauge transformations in the extended (to potentials and their derivatives) space are shown to lead to the Lie-Bäcklund tangent transformation group.

Let us first consider the simplest gauge theory—electrodynamics. The Maxwell equation

$$\begin{aligned} &\partial_\mu F_{\mu\nu} = 0 \\ &F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \end{aligned} \tag{1}$$

is a well known invariant under the gauge (gradient) transformation

$$A_\mu \to A_\mu + \partial_\mu \varphi(x). \tag{2}$$

(We give all expressions for the Euclidean space, but the obtained results are valid also for the Minkowski space.) In the extended space $(x_\mu,\ A_\alpha,\ A_{\alpha,\beta},\ A_{\alpha,\beta\gamma},\ldots)$ $(A_{\alpha,\beta} = \partial A_\alpha/\partial x_\beta$, one can write the transformation corresponding to (2):

$$A_\mu \to A_\mu + \mathrm{d}_\mu F(x_\nu,\ A_\alpha,\ A_{\alpha,\beta},\ A_{\alpha,\beta\gamma},\ldots) \tag{3}$$

where d_μ is the total derivative

$$\mathrm{d}_\mu \equiv \frac{\partial}{\partial x_\mu} + A_{\alpha,\mu}\frac{\partial}{\partial A_\alpha} + A_{\alpha,\beta\mu}\frac{\partial}{\partial A_{\alpha,\beta}} + \ldots$$

(transformations (3) and (2) and obviously invariant).

Now let us recall that the system of differential equations

$$\omega_p(x_\mu,\ A_\alpha,\ A_{\alpha,\beta},\ A_{\alpha,\beta\gamma},\ldots) = 0 \qquad p = 1,\ldots,M$$

(x_μ and A_α are the arguments and functions respectively) admits a Lie-Bäcklund tangent transformation group generated by a Lie-Bäcklund operator

$$\begin{aligned} &X = f_\alpha\frac{\partial}{\partial A_\alpha} + (\mathrm{d}_\nu f_\alpha)\frac{\partial}{\partial A_{\alpha,\nu}} + (\mathrm{d}_\mu \mathrm{d}_\nu f_\alpha)\frac{\partial}{\partial A_{\alpha,\nu\mu}} + \ldots \\ &f_\alpha = f_\alpha(x_\mu,\ A_\beta,\ A_{\beta,\nu},\ A_{\beta,\nu\mu},\ldots) \end{aligned} \tag{4}$$

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(we write the corresponding tangent vector field of the group in the canonical form) if (and only if)

$$\begin{aligned} X\omega_p &= 0 \\ \omega_p &= 0 \\ \mathrm{d}_i\omega_p &= 0 \\ \mathrm{d}_j\mathrm{d}_i\omega_p &= 0 \\ &\vdots \end{aligned}$$

(where $\mathrm{d}_i\omega_p = 0$, $\mathrm{d}_j\mathrm{d}_i\omega_p = 0 \ldots$ are the differential consequences of the initial system). For details of the theory of Lie–Bäcklund transformations, see Anderson and Ibragimov (1979) and Ibragimov (1983).

Thus, the Maxwell equation (1) admits the group with the Lie–Bäcklund operator (4) with

$$f_\mu = \mathrm{d}_\mu \varphi \tag{5}$$

where $\varphi = \varphi(x_\nu, A_\alpha, A_{\alpha,\beta}, \ldots)$ is an arbitrary function. If $\varphi = \varphi(x)$, equation (5) defines the usual gauge transformation (2). Each U(1) gauge invariant system (depending on $F_{\mu\nu}$ only) evidently posseses the same property.

Now let us proceed to non-Abelian gauge theories and consider the Yang–Mills equation with an arbitrary simple gauge group G.

$$\begin{aligned} &\partial_\mu G^a_{\mu\nu} + g f_{abc} A^b_\mu G^c_{\mu\nu} = 0 \\ &G^a_{\mu\nu} = A^a_{\nu,\mu} - A^a_{\mu,\nu} + g f_{abc} A^b_\mu A^c_\nu \qquad a, b, c = 1, \ldots, N \end{aligned} \tag{6}$$

where N is the dimension of the group G and f_{abc} are the structure constants. The tensor f_{abc} is completely antisymmetric, due to the choosing of the invariant inner product of the basis generators orthonormal (in the adjoint representation):

$$(L_a, L_b) = K\,\mathrm{Tr}(L_a L_b) = \delta_{ab}.$$

Using the invariance of the Yang–Mills equation (6) under gauge transformations (in the infinitesimal and finite form)

$$\begin{aligned} &A^a_\mu \to A^a_\mu - \frac{1}{g}\partial_\mu \omega^a + f_{abc}\omega^b A^c_\mu \\ &A_\mu \to U A_\mu U^{-1} + \frac{\mathrm{i}}{g} U \partial_\mu U^{-1} \\ &A_\mu = A^a_\mu L_a \qquad U = \exp(-\mathrm{i}\omega^a L_a) \qquad \omega^a = \omega^a(x) \end{aligned} \tag{7}$$

we introduce the dependence of the group parameters on all the variables of the extended space x_μ, A^a_μ, $A^a_{\mu,\alpha}$, $A^a_{\mu,\alpha\beta}, \ldots$ (analogously to the Abelian case).

Then equations (7) change to

$$\begin{aligned} &A^a_\mu \to A^a_\mu - \frac{1}{g}\mathrm{d}_\mu \omega^a + f_{abc}\omega^b A^c_\mu \\ &A_\mu \to U A_\mu U^{-1} + \frac{\mathrm{i}}{g} \mathrm{d}_\mu U^{-1} \\ &\omega^a = \omega^a(x_\mu, A^b_\mu, A^b_{\mu,\alpha}, \ldots) \end{aligned} \tag{8}$$

(the invariance of equations (6) under transformations (8) can be checked easily). Thus, the Yang–Mills equation admits a group of Lie–Bäcklund transformations with the operator

$$X_\varphi = f^a_{\mu\varphi}\frac{\partial}{\partial A^a_\mu} + (\mathrm{d}_\alpha f^a_{\mu\varphi})\frac{\partial}{\partial A^a_{\mu,\alpha}} + (\mathrm{d}_\beta \mathrm{d}_\alpha f^a_{\mu\varphi})\frac{\partial}{\partial A^a_{\mu,\alpha\beta}} + \ldots$$

$$f^a_{\mu\varphi} = -\frac{1}{g}\mathrm{d}_\mu\varphi^a + f_{abc}\varphi^b A^c_\mu \tag{9}$$

$$\varphi^a = \varphi^a(x_\nu, A^b_\nu, A^b_{\nu,\alpha}, \ldots).$$

The commutation relation in the algebra of the Lie–Bäcklund operators is

$$[X_\varphi, X_\psi] = X_\theta \qquad \theta^a = X_\varphi\psi^a - X_\psi\varphi^a + f_{abc}\psi^b\varphi^c. \tag{10}$$

Here the Jacobi identity

$$f_{abn}f_{ncd} + f_{bcn}f_{nad} + f_{can}f_{nbd} = 0$$

has been used. Note that the Lie–Bäcklund algebra (9) for equation (6) is non-trivial: it cannot be obtained from the Lie point symmetry group of the Yang–Mills equation by a simple prolongation (e.g., Ovsyannikov 1978). Really, besides the local gauge invariance (7) the Yang–Mills equation possesses the group of conformal transformations (Mack and Salam 1969, Schwartz 1982). Therefore, by a simple prolongation the group of point transformations of the Yang–Mills equation evidently does not lead to the group with operators (9). (The same conclusion is valid also for the Maxwell equation.)

Similar Lie–Bäcklund operators can also be constructed for other gauge theories with interaction between different fields. For example, let us consider the gauge invariant (with an arbitrary simple group G of dimension N) system of Yang–Mills fields coupled with the multiplet of the N scalar particles ϕ^a:

$$L = -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \tfrac{1}{2}(D_\mu\phi^a)(D_\mu\phi^a) - V(\phi^2) \tag{11}$$

where the adjoint representation is chosen,

$$D_\mu\phi^a = \partial_\mu\phi^a + gf_{abc}A^b_\mu\phi^c \qquad a, b, c = 1, \ldots, N$$

and $V(\phi^2)$ is the G-invariant polynom with respect to ϕ^a (for the Higgs model $V(\phi^2)$ is the polynom of fourth order with the minimum at $\phi = v$ (e.g., Abers and Lee 1973)).

The local gauge transformations leaving Lagrangian (11) invariant, are given by expressions (7) with $(\omega^a = \omega^a(x))$

$$\phi^a \to \phi^a + f_{abc}\omega^b\phi^e$$

$$\phi^a \to (U)_{ab}\phi^b.$$

As earlier, let us allow for the dependence of the group parameters ω^a on all the variables of the extended space:

$$\omega^a = \omega^a(x_\mu, A^b_\mu, A^b_{\mu,\alpha}, \ldots, \phi^b, \phi^b_{,\alpha}, \ldots) \tag{12}$$

(the invariance of Lagrangian (11) holds as well).

The Lie-Bäcklund operator for system (11) has the form

$$X_\omega = f^a_{\mu\omega}\frac{\partial}{\partial A^a_\mu} + (\mathrm{d}_\alpha f^a_{\mu\omega})\frac{\partial}{\partial A^a_{\mu,\alpha}} + \ldots$$
$$+ \bar{f}^a_\phi \frac{\partial}{\partial \phi^a} + (\mathrm{d}_\alpha \bar{f}^a_\phi)\frac{\partial}{\partial \phi^a_{,\alpha}} + \ldots \qquad (13)$$

where

$$f^a_\omega = -\frac{1}{g}\,\mathrm{d}_\mu \omega^a + f_{abc}\,\omega^b A^c_\mu$$
$$\bar{f}^a_\omega = f_{abc}\,\omega^b \phi^e$$

and ω^a is an arbitrary function (12). As in the case of the pure Yang-Mills theory, the commutator of two Lie-Bäcklund fields has the same form (10) in space (12).

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